New exact solutions of relativistic hydrodynamics

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Abstract. A new class of simple and exact solutions of relativistic hydrodynamics is presented, and the consequences are explored in data analysis. The effects of longitudinal work and acceleration are taken into account in an advanced estimate of the initial energy density and the life-time of the reaction.

1. Introduction: High temperature superfluidity

Fluid dynamics is based only on local conservation laws and thermodynamics, so it can be successfully applied to a vast range of physical phenomena. Recently, a new state of matter has been created in Au+Au collisions at RHIC, and, surprisingly, it was found to flow as a perfect fluid [1]. The kinematic shear viscosity of this nearly perfect fluid has been determined and found to be at least a factor of 4 smaller, than that of the superfluid ^{4}He [2]. Let us refer to this property as *high temperature superfluidity*, to highlight that the fluid of quarks in the super-high, $T \simeq 2$ terakelvin temperature range flows better than the superfluid ^{4}He in the extremely low, 1-4 K temperature region.

Exact solutions of perfect fluid hydrodynamics have powerful implications in high energy particle and nuclear physics. Although the renowned Landau-Khalatnikov solution [3, 4, 5] is only implicitly given, it describes well the energy dependent increase of the width of the rapidity distribution. Although the renowned Hwa-Bjorken solution [6] lacks acceleration and yields a too perfectly flat rapidity distribution, it yields a key estimate of the initial energy density in high energy reactions. Other families of exact solutions were born from the desire of understanding the dynamics of high-energy heavy reactions [7, 8, 9, 10, 11, 12, 13, 14]. Here we present a new, accelerating, analytic, exact and explicit solution of relativistic hydrodynamics, which fits dn/dy data at RHIC and allows for an advanced estimate of initial energy density and life-time of the reaction.

2. Simple solutions of relativistic hydrodynamics

We discuss spherical (and one-dimensional) solutions, where r denotes the radial (or the single spatial) coordinate, and d is the number of dimensions. The pressure is denoted by p, the energy density by ε, and the temperature by T, the four-velocity

Case:	λ	d	κ	ϕ_λ
(a)	2	$\in \mathbb{R}$	d	0
(b)	$\frac{1}{2}$	$\in \mathbb{R}$	1	$\frac{\kappa+1}{\kappa}$
(c)	$\frac{3}{2}$	$\in \mathbb{R}$	$\frac{4d-1}{3}$	$\frac{\kappa+1}{\kappa}$
(d)	1	$\in \mathbb{R}$	$\in \mathbb{R}$	0
(e)	$\in \mathbb{R}$	1	1	0

Table 1. New exact hydrodynamical solutions are given by lines (a, b, c) and (e), while case (d) is the already known Hwa-Bjorken (and Hubble) solution.

by $u^\mu = \gamma(1, \mathbf{v})$, and $\mathbf{v} = v\mathbf{n}$ is the three-velocity. For the equation of state (EoS), we assume that $\varepsilon - B = \kappa(p + B)$, with $\kappa = 1/c_s^2$, where c_s is the speed of sound and B stands for a bag constant (that may have a vanishing value too). In high energy collisions, the entropy density σ is large, but the net charge density is small. Thus we assume that all the conserved charges have zero chemical potential. In perfect fluids, σ and four-momentum tensor $T_\nu^\mu = (\varepsilon + p)u^\mu u_\nu - p\delta_\nu^\mu$ are locally conserved: $\partial_\nu(\sigma u^\nu) = 0$, $\partial_\nu T^{\mu\nu} = 0$. With projections, we obtain the Euler and the energy equations:

$$(\varepsilon + p)u^\nu \partial_\nu u^\mu = \left(\delta_\rho^\mu - u^\mu u_\rho\right) \partial^\rho p, \tag{1}$$

$$(\varepsilon + p)\partial_\nu u^\nu + u^\nu \partial_\nu \varepsilon = 0. \tag{2}$$

Our solutions can be written down in the so-called Rindler-coordinates τ and η: $t = \tau \cosh\eta$, $r = \tau \sinh\eta$. We found the following expression for p and v:

$$v = \tanh \lambda\eta, \quad p + B = p_0 \left(\frac{\tau_0}{\tau}\right)^{\lambda d \frac{\kappa+1}{\kappa}} \left(\cosh \frac{\eta}{2}\right)^{-(d-1)\phi_\lambda}. \tag{3}$$

We obtain solutions in some special cases of the constants listed in Table I, and for any B. In what follows, we set $B = 0$ to simplify the description. While case (d) is the Hwa-Bjorken solution, case (a) is a new solution valid in arbitrary dimensions. It has uniformly accelerating trajectories. Case (b) and (c) were found jointly by T. S. Biró [15] (for $d = 1$) and by us (for any d). In these solutions p is finite in η. Case (e) has a special EoS and is valid only in 1+1 dimensions, but λ can be any real number. This is why we shall use it in the following applications.

3. Applications to high-energy reactions

For the description of high-energy reactions, we calculated the rapidity distribution, $\frac{\mathrm{d}n}{\mathrm{d}y}$ from the solution denoted as case (e) in Table I. Assuming sudden freeze-out at temperature $T(\tau_f, \eta = 0) = T_f$ on a hypersurface pseudo-orthogonal to $u^\mu(x)$, with a saddle-point integration in η and the transverse mass m_T (which becomes exact if $m/T_f \gg 1$, where m is the particle mass), we got the following formula [20]:

$$\frac{\mathrm{d}n}{\mathrm{d}y} \approx \left.\frac{\mathrm{d}n}{\mathrm{d}y}\right|_{y=0} \cosh^{-\frac{\alpha}{2}-1}\left(\frac{y}{\alpha}\right) \exp\left\{-\frac{m}{T_f}\left[\cosh^\alpha\left(\frac{y}{\alpha}\right) - 1\right]\right\}, \tag{4}$$

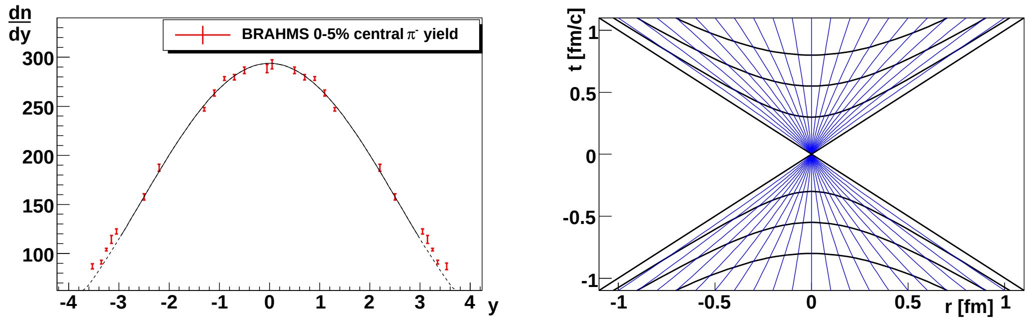

Figure 1. Left panel: Result of the fit of eq. (4) to BRAHMS $\frac{\mathrm{d}n}{\mathrm{d}\eta}$ data of Ref. [16]. Right panel: Flow trajectories and possible freeze-out hypersurfaces of the $\lambda = 1.18$ solution, which fits to BRAHMS $\frac{\mathrm{d}n}{\mathrm{d}y}$ data. (The physical solution is in the future light cone, but for aesthetic reasons its extension to the past light cone is also shown).

with $\alpha = \frac{2\lambda-1}{\lambda-1}$. The parameter $\Delta y^2 = \frac{\alpha}{m/T_f+1/2+1/\alpha}$ characterizes this distribution: it has a minimum at $y = 0$, if $\Delta y^2 < 0$, it is flat if $\Delta y^2 = 0$, (this is the case when $\lambda = 1$), otherwise it is nearly Gaussian. We extracted the λ parameter for collisions at $\sqrt{s_{NN}} = 200$ GeV by fitting eq. (4) to $\frac{\mathrm{d}n}{\mathrm{d}y}$ data measured by the BRAHMS collaboration [16]. Fig. 1 shows the fit, and the corresponding flow profile. We found $\lambda = 1.18 \pm 0.01$, which indicates the presence of acceleration, as shown in Fig 1 by the curvature of the fluid lines. This influences both the estimation of initial energy density (because of the faster expansion and the work done by the matter), and the estimation of the life-time of the reaction. This way we found a new estimate of the initial energy density ε_c that generalizes Bjorken's estimation, ε_{Bj} to non-flat rapidity distributions:

$$\frac{\varepsilon_c}{\varepsilon_{Bj}} = (2\lambda-1)\left(\frac{\tau_f}{\tau_0}\right)^{\lambda-1} \qquad \text{where} \quad \varepsilon_{Bj} = \frac{\langle m_t \rangle}{(R^2\pi)\tau_0}\frac{dn}{dy}\Big|_{y=0}. \tag{5}$$

The life-time estimation from the longitudinal length of homogeneity, R_{long} of Sinyukov and Makhlin, τ_{Bj} was also based on the Hwa-Bjorken solution [18]. Our new estimation, τ_c, which takes acceleration effects into account, is

$$R_{\text{long}} = \sqrt{\frac{T_f}{m_t}}\frac{\tau_c}{\lambda} \qquad \text{thus} \quad \tau_c = \lambda\tau_{Bj} = \lambda\sqrt{\frac{m_t}{T_f}}R_{\text{long}}. \tag{6}$$

The eqs. (5) and (6) show that both the initial energy density and the life-time of the reaction is under-estimated by formulas based on the accelerationless of Hwa-Bjorken solution. For more physical EoS than that super-hard $\kappa = 1$ corresponding to the $\lambda = 1.18$ solution, we conjectured new, advanced estimations of the initial energy density $\varepsilon_{c_s^2}$ and the life-time of the reaction $\tau_{c_s^2}$ as

$$\frac{\varepsilon_{c_s^2}}{\varepsilon_{Bj}} = (2\lambda-1)\left(\frac{\tau_f}{\tau_0}\right)^{\lambda-1}\left(\frac{\tau_f}{\tau_0}\right)^{(\lambda-1)(1-c_s^2)}, \tag{7}$$

$$\tau_{c_s^2} = [\lambda + (\lambda-1)(1-c_s^2)]\tau_{Bj}. \tag{8}$$

For their detailed explanation, see Refs. [19, 20]. Using the Bjorken estimate of $\varepsilon_{Bj} = 5$ GeV/fm^3 as given in Ref. [17], and $\tau_f/\tau_0 = 8 \pm 2$ fm/c, our analytic formula (5) yields

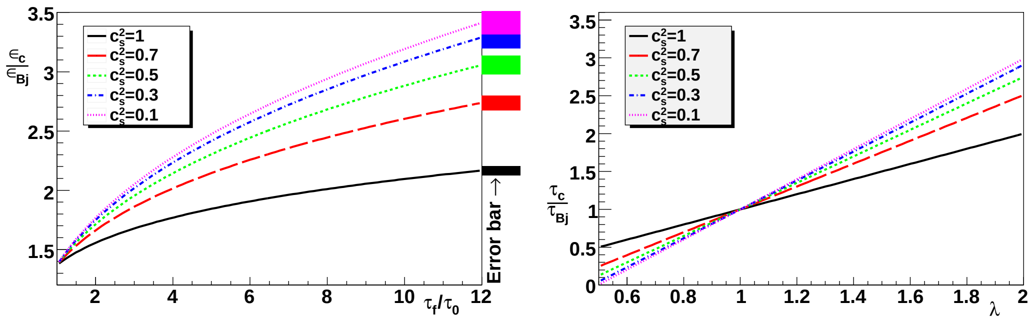

Figure 2. Dependence of $\varepsilon_{c_s^2}/\varepsilon_{Bj}$ on τ_f/τ_0 (left panel) and $\tau_{c_s^2}/\tau_{Bj}$ on λ (right panel) for various c_s^2 values. Solid lines for $c_s^2 = 1$ indicate the exact result of eqs. (5) and (6), the others show the conjectures of eq. (7).

$\varepsilon_c = (2.0 \pm 0.1)\varepsilon_{Bj} = 10.0 \pm 0.5$ GeV/fm^3 for $\lambda = 1.18 \pm 0.01$. For the life-time, eq. (6) implies a 18±1% increase. Fig. 2 shows the speed of sound dependence of $\varepsilon_{c_s^2}/\varepsilon_{Bj}$ as a function of τ_f/τ_0, and that of $\tau_{c_s^2}/\tau_{Bj}$ as a function of λ. For a realistic equation of state, where $c_s^2 = 0.1$ in $\sqrt{s_{NN}} = 200$ GeV Au+Au reactions [23], acceleration increases the initial energy density estimate to 15 GeV/fm^3, and the life-time estimate by 36 %.

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